NUMERICAL SOLUTION OF RATIONAL EXPECTATIONS MODELS WITH AND WITHOUT STRATEGIC BEHAVIOUR

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ABSTRACT

The assumption of forward-looking agents in theoretical macroeconomic models has become increasingly popular in recent years. Despite this, the implementation of forward-looking expectations in large econometric models has been slower to emerge. The purpose of this paper is to survey, in a non-technical manner, recent algorithms that have been developed to solve medium-size models when some agents in the models are assumed to have rational expectations. With an intuitive understanding of the algorithms, it is hoped that the technical source literature will be more readily accessible to model builders.

The use of game theory in macroeconomics has also seen a resurgence. The second part of this paper develops an algorithm which is useful for solving rational expectations models and can also be used to solve dynamic games between agents with forward-looking expectations. Although derived for a specific application, the algorithm is sufficiently general to be useful for solving a range of non-cooperative games.

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1. Introduction

The assumption of "model-consistent" or "rational" expectations, first implemented by John Muth (1961), has important implications for the solution of macroeconomic models. In models containing rational agents[1], current variables depend on the expected path of future variables[2]. Until recently, this assumption was only used in small analytical models because of the difficulty in solving models under rational expectations. Advances in computing power and technique have now allowed economy-wide and multi-country models to be solved[3]. There still appears to be an aversion to applying the available numerical techniques perhaps due to the inaccessibility of the source literature to many economists. With the goal of making the literature more accessible, this paper surveys the major numerical techniques which have been developed to solve large models, where expectations of some agents in the model are assumed to be formed rationally.

Section 2 of this paper examines the key implications of assuming model consistent expectations using a very simple model of exchange rate dynamics. This simple model introduces the concepts used extensively in section 3 and provides a useful introduction to the problem to be solved numerically. In section 3, alternative numerical techniques are discussed, focussing on the intuition behind the formal derivations presented in the literature. These techniques are the general analytical solution of Blanchard and Kahn (1980), the Multiple Shooting algorithm of Lipton et al (1983), the Fair-Taylor algorithm (Fair and Taylor (1983)) and a fourth algorithm developed by the author in joint work with Jeffrey Sachs on the MSG model.

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1. That is, agents who use all information available in deciding on actions and do not make any systematic errors.

2. See Begg (1982), Sheffrin (1983) and Taylor (1985) for surveys of the use and relevance of the rational expectations assumptions.

3. See for example the MSG model (McKibbin and Sachs (1986)), the Liverpool Model (Minford (1985)), the Taylor Model (Taylor (1986)) and Minimod (Haas and Masson (1986)).

The goal of section 3 is to present the techniques in a way that makes the algorithms more transparent than the original articles. Section 4 introduces a technique developed with Jeffrey Sachs that extends the algorithm in Oudiz and Sachs (1985) and moves the discussion of solving rational expectations models to the case where agents interact in a strategic manner.

2. A Simple Illustration of the Problem of Solving Rational Expectations Models

Consider a small open economy which is described by the following equations:

$$m_t - p_t = \alpha q_t - \beta i_i \tag{2.1}$$

$$q_t = \gamma(e_t + p_t^* - p_t) - \delta i_t \tag{2.2}$$

$$i_t = i_t^* + {}_t e_{t+1} - e_t \tag{2.3}$$

All variables are in logs. Starred variables are foreign variables. Equation (2.1) is the LM curve for the economy. Money demand (m-p) is a function of output (q) and the nominal interest rate (i). Prices are assumed to be sticky. This implies that the price level is not expected to change so that real and nominal interest rates are equal. Equation (2.2) is the IS curve for the economy. Aggregate demand is a positive function of the real exchange rate and a negative function of the interest rate. The nominal exchange rate is defined as the home price of foreign exchange so a rise in e is a depreciation of the exchange rate. Equation (2.3) gives the relation between domestic and foreign interest rates. It assumes that capital is perfectly mobile internationally and foreign and domestic bonds are perfect substitutes and therefore uncovered interest parity holds; domestic and foreign interest rates are equalised, adjusted for any expected exchange rate changes. The notation ${}_t e_{t+1}$ is used here to indicate the expectation formed in period t of e_{t+1}. We assume that agents form these expectations rationally, meaning that in a statistical sense, agents use the best linear unbiased predictor of the exchange rate. This is given as:

$${}_t e_{t+1} = E\{e_{t+1} | I_t\}$$

where the expectation operator E is conditional on the information set I_t which contains realizations on all endogenous and exogenous variables as of period t. In this case there is no uncertainty in the model so the assumption here is that of perfect foresight:

$$ {}_te_{t+1} = e_{t+1} $$

Prices are assumed to be sticky and the money supply (m) and foreign variables are assumed to be exogenous. The model therefore has three equations and three unknowns: q, i and e. Notice that to solve the model in period t we need to know e_{t+1}. This is the problem to be examined in the rest of this section.

A solution for period t can be found by substituting (2.2) into (2.1) to solve for i.

$$ i_t = f(e_t+p_t^* - p_t) + g(m_t-p_t) $$

where $f = \frac{\alpha\gamma}{\alpha\delta+\beta}$ $g = \frac{1}{\alpha\delta+\beta}$

We can rewrite (2.3) as

$$ e_t = e_{t+1} + i_t^* - i_t $$

substituting for i and stacking all exogenous variables m_t, p_t, p_t^*, i_t^* in a matrix Z_t we find:

$$ e_t = h\, e_{t+1} + j\, Z_t \qquad (2.4) $$

where $h = (\alpha\delta+\beta)/(\alpha\delta+\beta+\alpha\gamma) < 1$

and j is a vector of coefficients on the exogenous variables contained in the vector Z.

The problem now is that the solution for e depends on the expectation of e_{t+1}. To find this expectation which is based on the known structure of the model, we can now lead the model by one period and take the expectation of e_{t+1}:

$$e_{t+1} = he_{t+2} + jZ_{t+1}$$

Substituting this into (2.4) gives e_t as a function of the exogenous variables in periods t and t+1 as well as a function of the expectation formed in period t of e_{t+2}. For convenience, assuming that the path of future exogenous variables is constant, we can repeat this procedure to find:

$$e_t = h^k e_{t+k} + j(1-h^{k+1})/(1-h) \quad \bar{Z} \qquad (2.5)$$

One problem with this solution is that, without any other restrictions, we can pick any value of the expected exchange rate in period t+k, to give a solution for the current exchange rate. Expectations can be self-fulfilling! Generally the problem is solved in one of two ways. In the case where the model has been derived by solving a dynamic optimisation problem, terminal conditions will be available as part of the solution to the problem. Where this is not the case, the usual procedure is to take the stable solution as the solution to the model.

Several points can be made about the solution given in (2.5). In the one dimensional case which we are examining, 1/h corresponds to the eigenvalue of the model. It can be seen that $h<1$ which implies that the eigenvalue is greater than unity. The model is therefore fundamentally unstable. In the linear case there is one unique initial value of e_t which prevents the model from exploding over time. In this case it is where $e_t=j/(1-h)\bar{Z}$. The non-exploding path for e is called the stable manifold of the model. The object of the numerical algorithms is to find this unique stable path or, equivalently, to find a unique initial value for the vector e in a multidimensional context. In the case of non-linear models the initial value and the path are not necessarily unique. The conditions for uniqueness in the linear case are derived rigorously in Blanchard and Kahn (1980). There is a unique solution if the number of jumping variables (i.e. variables such as e which jump in response to news) equals the number of eigenvalues outside the unit circle.

Another important point to note about equation (2.5) is that the effect on e_t of future values of the expected variable diminishes over time. The rate of decay of the influence of the future values depends again on the eigenvalue of the model. This point is very important in practice because in all solution techniques some assumption must be made about terminal values of variables. It implies that errors in initial guesses of terminal conditions will have negligible effects on final results, if the terminal period is sufficiently far in the future. Whether or not the terminal period is "sufficiently far in the future", is a function of the eigenvalues of the system.

This section has given an introduction into the nature of the problem we wish to solve. The next section will develop the themes raised here, for larger systems of equations.

3. Numerical Solutions of Rational Expectations Models

This section considers four numerical techniques that solve the multidimensional problem. These are the Blanchard-Kahn solution, Multiple Shooting, Fair-Taylor and the MSG techniques. The multiple shooting and Fair-Taylor techniques are specifically designed to solve the general non-linear problem. For clarity we will concentrate the discussion in the framework of a linear model.

At this point it is worth introducing some terminology.

A model written in minimal state-space representation is in the form:

$$\bar{X}_{t+1} = z_1 \bar{X}_t + z_2 E_t \qquad (3.1)$$

where $\bar{X}$ is a mx1 vector of evolving variables.

E is a sx1 vector of exogenous variables.

We can partition the $\bar{X}$ matrix in a convenient way. Let the first n variables be state or predetermined variables whose values are inherited from the past and the remaining m-n variables be "jumping variables". Jumping variables are variables over which agents form expectations and which can change in response to new information in period t. The model (3.1) can be rewritten as:

$$X_{t+1} = a_1X_t + a_2e_t + a_3E_t \tag{3.2}$$

$$_te_{t+1} = b_1X_t + b_2e_t + b_3E_t \tag{3.3}$$

where X is a vector of state variables whose value is inherited from the past evolution of the system.

e is a vector of jumping variables, determined within the current period by the structure of the model and information about current and all future variables.

E is a vector of exogenous variables (including policy instruments).

This model could easily be solved forward as in the case of standard difference equations given X_0, e_0 and a path for E. That is, given the initial values for the state and jumping variables we can solve forward for X_1 and e_1 and so forth. The problem faced in attempting to numerically solve a rational expectations model is that there are only initial values for the set of state variables X_0 and terminal values for the set of jumping variables e_T or e_∞ (either assumed or from some optimisation solution). To solve the model in period 0, and for every period until T, requires knowledge of e_0 which requires knowledge of the solution in period 1 and so forth. This is called a two-point boundary value problem. Two points on the equilibrium path of the economy are known and these are both needed to define the path between them. Analytical solutions to models containing rational expectations can be found in simple cases by using techniques which solve these types of two-point boundary value problems such as illustrated in section 2. That is, to solve the model requires use of restrictions provided by initial values of state variables and some terminal conditions on jumping variables. The terminal values can be given as a fixed value in some finite terminal period or, in an infinite horizon problem, by a tranversality condition which imposes that in the infinite limit a variable is bounded.

Numerical solutions to more complex systems have been slower to emerge although there are now several techniques commonly used. The solutions provided by these techniques to be discussed can be better understood with the aid of Figure 1. Suppose that the multidimensional system of state and jumping variables can be compressed into points in a two-dimensional space. Each point such as A consists of a set of values for each of the state and jumping variables in the model $\{X_0, e_0\}$. Suppose point B summarises the terminal value for the problem which gives $\{X_T, e_T\}$. The problem is to find the unique path between A and B. We have initial values for X_0 and terminal values for e_T.

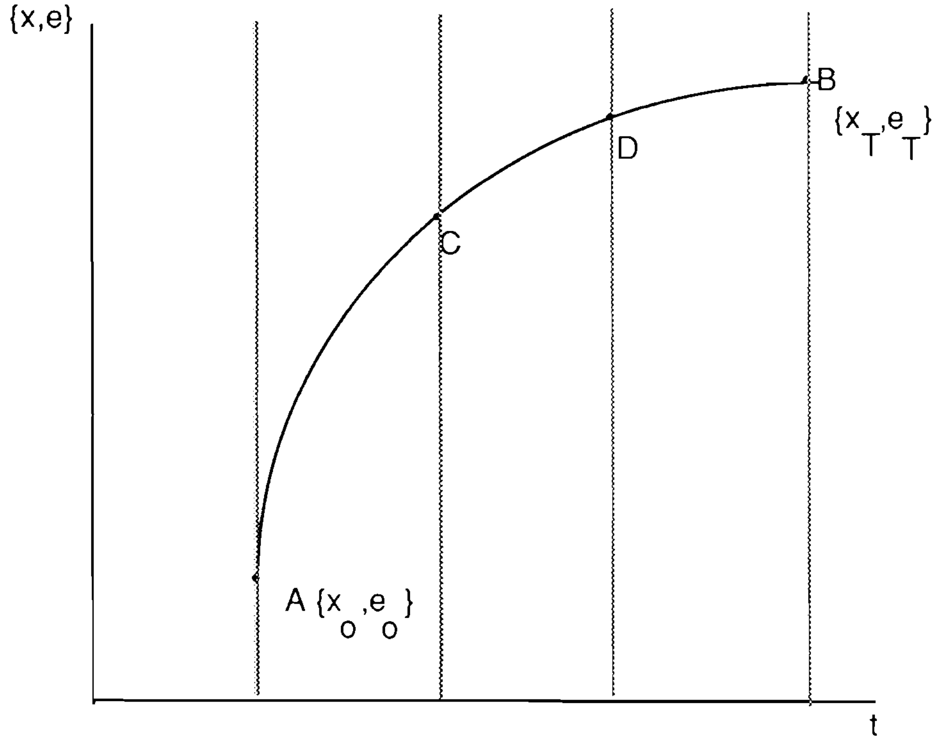
a. Blanchard-Kahn Solution

For a linear system, a general analytical solution is provided by Blanchard and Kahn (1980). This technique is a generalisation of the solution to a difference equation system as derived in section 2 above. It is based on transforming the transition matrix z_1 in (3.1) into its eigenvalue and eigenvector matrices. A solution is obtainable using the Blanchard-Kahn technique if the number of eigenvalues outside the unit circle is equal to the number of jumping variables. A more general analytical solution is provided by Chow and Reny (1984) but will not be discussed here.

b. Multiple Shooting Algorithm

In the case of non-linear systems, the technique of multiple shooting has been applied to the economics literature by Lipton, Poterba, Sachs and Summers (1983). The shooting technique can be described intuitively as follows. Initial values are assumed for the jumping variables $\bar{e}_0$. The model is then solved forward until the terminal period (or some finite period that is considered a good approximation to the infinite horizon) is reached. The terminal conditions on the jumping variables e_T, are then compared to the solution values for the jumping variables $\bar{e}_T$. If these are not equal, the initial guesses of the jumping variables are updated using some error correction procedure (i.e. Newton's Method). In the case of multiple shooting, the solution interval is divided into sub-intervals, such as shown in figure 1. The object is to solve the model to pass through intermediate points C and D. With the aid of auxiliary variables, the model is then

Figure 1: Illustration of Multiple Shooting Algorithm



solved, shooting within each sub-interval until convergence of the model solution to the terminal conditions is reached. A problem with this algorithm is that each sub-interval increases the dimensionality of the system to be solved.

c. Fair-Taylor Algorithm

Fair and Taylor (1983) have also developed a technique which has become popular because it tends to find a solution to models at a much lower cost than the multiple shooting algorithm. In the Fair-Taylor technique, an arbitrary terminal period, T, is chosen. Equation (3.3) is rewritten

$$e_t = b_2^{-1}({}_t\bar{e}_{t+1} - b_1 X_t - b_3 E_t) \qquad (3.4)$$

where ${}_t\bar{e}_{t+1}$ is taken as an exogenous guess for the expectation of e.

The paths of expected variables $\{{}_te_1, \ldots, {}_te_T\}$ are guessed and the model is solved assuming these expectations. The solution paths for the expected variables are then compared to the guesses and the guesses are updated using an error correction method. This iterative procedure is repeated until the expected path equals the actual path. The terminal period is then extended and the procedure repeated until the terminal period choice has no effect on the solution path. In our experience, the iteration on terminal period (called type III iteration in Fair and Taylor) is only required at the initial stage of simulating the model. Once a period length is established it rarely needs to be updated.[4]

d. MSG Algorithm

A fourth solution procedure is used by McKibbin and Sachs in solving the MSG model. It is developed in the next section in more detail. The first step is

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4. A copy of this algorithm which solves up to 90 simultaneous equations on a PC is available from Aptech System, P.O. Box 6487 KENT WA 98064, USA. The algorithm is written in GAUSS.

to linearize the model around an assumed steady state. Assume that in any period, the jumping varables (e) can be written as a function of the inherited state variables (X) current exogenous variables (E), and the future path of exogenous variables, in the following form:

$$e_t = H_1X_t + H_2E_t + C_{5t}\ \{E_{t+1}\ldots.E_T\}$$

To find the matries H_1, H_2 and C_{5t} an iterative technique is used which essentially solves the model backwards. First the model is solved in an arbitrary terminal period T. By assuming period T is the last period, the future expectations of variables beyond period T are irrelevant. The model can then be rewritten in period T as:

$$e_T = H_{1T}X_T + H_{2T}\ E_T$$

where the H_1 and H_2 matricies are time subscripted.

Moving back to period T-1, the future jumping variables (e_T) are solved as functions of state variables (which are determined in period T-1) and exogenous variables. The model can now be rewritten in period T-1 given the rule for e_T. This procedure is repeated, solving backwards until the matricies H_1 and H_2 become independent of the terminal period chosen. This process also generates a cumulation rule for all future exogenous variables which is summarized in the matrix C_{5t}.

Once the rules linking e_t to X_t and current and future exogenous variables are found, the model can be solved in any period t using the additional condition:

$$e_t=H_1\ X_t+H_2E_t+C_{5t}.$$

This enables the model to be solved forward as a standard difference equation system. This technique is numerically equivalent to the Blanchard-Kahn solution although it is very convenient for implementing dynamic game theory which is discussed below.

4. A Numerical Algorithm that Allows for Strategic Behaviour

There is now a large literature on the question of time consistent policy optimisation in both closed and open economies when agents are forward looking. In this section the MSG algorithm is developed further to allow for explicit optimisation of an objective function in a rational expectations model.

The algorithm can be used for many different applications: an economy of rational atomistic agents in which a government optimises an objective function; an economy in which a government and union strategically optimise separate objective functions; two or more open economies in which governments optimise objective functions strategically in a cooperative or non-cooperative manner, with and without strategic interactions with private agents.

The algorithm is developed in terms of strategic interactions between governments in a two-country world. It is based on the dynamic programming algorithm for dynamic games developed in Sachs and Oudiz (1985).

Put simply, the idea is to find a set of feedback rules for government policy (or control variables) from the optimisation of an objective function, where the rules link policy to the current state variables. Associated with the solution, we need to find the current jumping variable also as a function of current state variables. Once we find the rules, then given initial state variable (X_0), we can determine initial jumping variables (e_0) and therefore we can find a solution to the model. This is discussed more rigorously below.

Consider a general system of equations summarised by:

$$X_{t+1} = \phi_1(X_t, e_t, U_t, E_t, Z_t) \tag{4.1}$$

$$Z_t = \phi_2(X_t, e_t, U_t, E_t, Z_t) \tag{4.2}$$

$${}_te_{t+1} = \phi_3(X_t, e_t, U_t, E_t, Z_t) \tag{4.3}$$

$$\tau_t = \phi_4(X_t, e_t, U_t, E_t, Z_t) \qquad (4.4)$$

where X_t is a vector of state variables

e_t is a vector of jumping variables (such as forward looking asset prices)

U_t is a vector of control variables (such as monetary and fiscal policies)

τ_t is a vector of target variables which can include state, jumping or control variables

E_t is a vector of exogenous variables

Z_t is a vector of endogenous variables that do not effect the dynamics of the system

18e initial step of the solution technique is to linearise this system around some point, usually either the steady state or a point on the transition path, using a first order Taylor approximation. The impact of the linearisation on the results will depend on several factors. Firstly, if steady state consequences of policies are to be examined then linearising may cause problems. Also, if the model is highly non-linear then the effect of the future path of the economy on the current period may be distorted.

In practice, we have found very little difference between the linear and non-linear versions of the MSG model when examining short-run properties.

Finding numerical derivatives and solving for the endogenous variables (Z) as a function of the other variables in the system, the model can be written:

$$\bar{X}_{t+1} = a_1\bar{X}_t + a_2\bar{e}_t + a_3\bar{U}_t + a_4\bar{E}_t \qquad (4.5)$$

$$_t\bar{e}_{t+1} = b_1\bar{X}_t + b_2\bar{e}_t + b_3\bar{U}_t + b_4\bar{E}_t \qquad (4.6)$$

$$\tau_t = \gamma_1\bar{X}_t + \gamma_2\bar{e}_t + \gamma_3\bar{U}_t + \gamma_4\bar{E}_t \qquad (4.7)$$

where each of the coefficient matrices (a_1, b_2, γ_3, etc.) are numerical derivatives evaluated at the point of linearisation[5] and a bar

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5. e.g. $a_1 = \partial X_{t+1}/\partial X_t$ evaluated at X_0.

over a variable is the deviation of a variable from the point of linearisation. To avoid excessive notation, the bars over variables will be dropped with the understanding that all future references will be to variables as deviations from some level.

The final assumption added to the above system is that agents take into account all available information in forming expectations about future variables. Agents have rational or model consistent expectations which implies that their expectations of future variables are correct on average. In the current paper we assume perfect foresight so the assumption is:

$$ {}_{t}e_{t+1} = E\{e_{t+1} | I_t\} = e_{t+1} \tag{4.8} $$

where a subscript t before a variable indicates the expectation of that variable taken in period t based on the information available in that period.

Now introduce optimising policy-makers. Assume that policy-makers choose the control variables (U) to maximise an intertemporal utility function:

$$ W_t = \Sigma_{t=0}^{\infty} \delta^t \mu(\tau) \tag{4.9} $$

subject to the structure of the economy given in (4.5) to (4.8). When the social welfare function is not explicitly a quadratic loss function, it is made quadratic by linearising using the first two terms of a Taylor's series expansion. The problem for country i becomes to choose a vector of control variables U_{it}, to maximise:

$$ W_{it} = \Sigma_{t=0}^{\infty} \delta^t \{\Omega_1 \tau_{it} - \tau_{it}' \Omega_2 \tau_{it}\} \tag{4.10} $$

subject to:

$$ X_{t+1} = a_1 X_t + a_2 e_t + a_3 U_t + a_4 E_t \tag{4.11} $$

$$ e_{t+1} = b_1 X_t + b_2 e_t + b_3 U_t + b_4 E_t \tag{4.12} $$

$$\tau_{it} = \gamma_{1i}X_t + \gamma_{2i}e_t + \gamma_{3i}U_t + \gamma_{4i}E_t \tag{4.13}$$

where matrices related to control variables are stacked in the following way:

$a_3 = [a_{31}, a_{32}, \ldots a_{3j}]$ for j countries

$$U_t = \begin{vmatrix} U_{1t} \\ U_{2t} \\ U_{jt} \end{vmatrix}$$

In the case where the system summarises more than one country (or more than one strategic player), several different assumptions can be made. For example, each policy-maker can be assumed to undertake the optimisation taking as given the policies of other governments. This is the Nash-Cournot equilibrium of the dynamic game and is the equilibrium used here to represent non-cooperative behaviour between governments. An alternative is to assume that a central planner undertakes the optimisation of some weighted combination of the two countries' welfare function. This can then be considered the case of cooperation. Other assumptions are possible such as one country or group of countries acting as Stackelberg leaders in formulating policy. These other equilibrium concepts are not explored further here.

Within the class of equilibria considered here, there are various solutions possible depending on the constraints placed on policy-makers by such issues as time consistency and credibility. One solution is to undertake the maximisation of (4.10) in period t and find a path for policy taking as given the expectations of private agents. This is the optimal control solution. Kydland and Prescott (1975) point out that in a model with forward looking agents, the government finding an optimal control solution to a problem in period t will generally find it optimal in period t+1 to deviate from the pre-announced path. The optimal control solution does not satisfy Bellman's criterion for optimality. Once private agents have made decisions based on the announced policy, the problem changes. In a repeated game the pre-announced rule is no longer credible unless the present government can make some form of binding commitment to follow the chosen path of time.

In addition to the issue of time consistency of policies, there is the issue of the form of the rule being followed. The government can choose the entire path of policy settings (open loop policy) or it can choose a rule for the control variables which depend on the realisations of state and exogenous variables (closed loop). Here the focus is on closed loop policies. We also focus on the time consistent policies since they are more likely to be observed in a deterministic world where credibility is difficult to establish.

As mentioned above, the case of optimising governments playing dynamic games and exogenous policy shifts in forward looking models, can both be handled by the same solution technique. The problem is solved in this paper by a technique of dynamic progamming.[6] The technique proposed here is to first solve a finite period optimisation problem where the terminal period is arbitrarily chosen to be some period, T. Solving the problem in period T, gives a solution for the jumping and control variables in period T. The problem is then solved in period T-1, taking as given the policy rules being followed in the next period and the state variables inherited. The forward looking variables are then conditioned on the known future rules. The rules which are found for the finite period problem will be time dimensioned. The second step of the procedure is to find the limit of the finite period problem as $T \to \infty$. The limit is found by repeating the backward recursion procedure until rules are found for the control variables and the jumping variables which do not change as the terminal period is moved further away. The case where policy-makers are not optimising is found by setting the rule linking the control variables to the state and exogenous variables to an arbitrary rule or to zero (for no policy action) during the backward recursion. The rule for the jumping variables is therefore the unique stable manifold of the system. The uniqueness derives from the linearity of the system.

This is formally derived as follows.

Define the value function for any country i as:

$$V_{it} = \max_{U_{it}} \Omega_i \tau_{it} - \tau_{it}' \Omega_2 \tau_{it} + \delta V_{it+1}(X_{t+1}, C_{3t+1})$$

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6. See Oudiz and Sachs (1985) and Currie and Levine (1985) for similar solution techniques.

subject to (4.11) to (4.13) where C_{3t+1} is a constant containing the accumulated values of future exogenous variables.

In solving this problem we are trying to find matrices Γ_1 and Γ_2, of a linear policy rule:

$$U_t = \Gamma_1 X_t + \Gamma_2 E_t + C_{4t}$$

and matrices S_1, S_2, S_3, S_4 and S_5 such that:

$$V(X_t, C_{3t}) = S_{1i} X_t + S_{2i} E_t - X_t' S_{3i} X_t - 2X_t' S_{4i} E_t - E_t' S_{5i} E_t$$

where $$V_{it} = \underset{U_{it}}{\text{Max}}\ \Omega_i \tau_{it} - \tau_{it}' \Omega_2 \tau_{it} + \delta V_{it+1}(X_{t+1}, C_{3t+1})$$

subject to (4.11) to (4.13). We also need to find matrices H_1 and H_2 that ensure that the jumping variables adjust to keep the model on the stable manifold where:

$$e_t = H_1 X_t + H_2 E_t + C_{5t}$$

We know that the stable manifold can be expressed in this way due to the solution in Blanchard and Kahn (1980). The iterative technique which solves this problem begins by converting the infinite period problem into a finite period problem where the terminal period is some arbitrary period T. Assume that in period T+1, the jumping variables have stabilised and $V_{T+1}(X_{T+1}, C_{3T+1}) = 0$. This implies:

$${}_Te_{T+1} = e_T \ . \qquad (4.14)$$

Substituting (4.14) into (4.12) gives:

$$e_T = s_1 X_T + s_{2i} U_{iT} + s_3 E_T \ . \qquad (4.15)$$

The target variables can now be written as a function of state, control, and exogenous variables by substituting (4.15) into (4.13) to find:

$$\tau_{iT} = u_{1i}X_T + u_{2i}U_{iT} + u_{3i}E_T \tag{4.16}$$

This can be substituted into the welfare function (4.10) for period T and the problem written:

$$\underset{U_{iT}}{\text{Max}} \quad \Omega_1(u_{1i}X_T + u_{2i}U_{iT} + u_{3i}E_T) - (u_{1i}X_T + u_{2i}U_{iT} + u_{3i}E_T)'\Omega_2(u_{1i}X_T + u_{2i}U_{iT} + u_{3i}E_T)$$

Solving this single period problem, the first order condition for country 1 is:

$$u_{21}'\Omega_1 u_{1i}X_T + u_{21}'\Omega_1 u_{2i}U_{1T} + u_{21}'\Omega_1 u_{3i}E_T + C_1 = 0$$

This can be stacked for each country and rewritten:

$$MM_T U_T = -NN_T X_T - PP_T E_T + C_{1T}$$

or:

$$U_T = \Gamma_{1T}X_T + \Gamma_{2T}E_T + C_{4T} \tag{4.17}$$

Equation (4.17) gives a rule for the control variables as a function of the state and exogenous variables in period T conditional on the known future. This can be substituted into (4.15) to give a rule for the jumping variables conditional on the known government policy rule.

$$e_T = H_{1T}X_T + H_{2T}E_T + C_{5T} \tag{4.18}$$

where: $H_{1T} = s_1 + s_2\Gamma_{1T}$; $H_{2T} = s_3 + s_2\Gamma_{2T}$

and s_2 is a stacked matrix $[s_{2i}|s_{2j}]$ for each country i,j.

The rules for control variables and jumping variables given in (4.17) and (4.18) can be substituted into the equation for the target variables given in (4.13). This can then be substituted into the welfare function to find the value function in period T as a function of the state and exogenous variables in T as well as the constants.

$$V_{it} = f(X_T, E_T, C_{2T}) \tag{4.19}$$

Given the value function in each period and accumulating all future exogenous variables and constants into a constant C_3, we can solve the problem in any period t where the policy-maker is to select the vector of control variables, U_{it}, to maximise:

$$\Omega_1\tau_{it} - \tau_{it}'\Omega_2\tau_{it} + \delta V_{it+1}\{X_{t+1}, C_{3t+1}\} \tag{4.20}$$

subject to (4.11) to (4.13).

To solve this problem, note that we have V_{it+1} as a function of X_{t+1}. Using the equation of motion of the state variables given in (4.11), we can therefore write V_{it+1} as a function of period t variables. The entire problem faced in period t can now be written in terms of period t variables. Consider the specific steps in solving this problem. We have from the solution of the problem in period t+1:

$$e_{t+1} = H_{1t+1}X_{t+1} + C_{3t+1} \tag{4.21}$$

where

$$C_{3t+1} = C_{5t+1} + H_{2t+1}E_{t+1}$$

We can substitute the equation for the jumping variables (e_{t+1}) from (4.12) and the equation of motion for the state variables (X_{t+1}) given in (4.11) into (4.21). Simplifying gives:

$$e_t = s_1X_t + s_{2i}U_{it} + s_3E_t + s_{4t} \tag{4.22}$$

Equation (4.22) can again be substituted into the equation for the targets given in (4.13) to find:

$$\tau_{it} = u_{1i}X_t + u_{2i}U_{it} + u_{3i}E_t + u_{4it} \tag{4.23}$$

The optimization problem can now be written with the current target variables as a function of the state, control and exogenous variables and the value function V_{t+1} as a function of the current state, control and future exogenous variables.

$$\max_{U_{it}} \quad \Omega_i \tau_{it} - \tau_{it} \Omega_2 \tau_{it} + V_{it+1} (X_{t+1}, C_{3t+1})$$

subject to:

$$\tau_{it} = u_{1i} X_t + u_{2i} U_{it} + u_{3i} E_t + u_{4it}$$

and

$$X_{t+1} = a_1 X_t + a_2 (s_1 X_t + s_{2i} U_{it} + s_3 E_t) + a_{3i} U_{it} + a_4 E_t$$

The rewritten problem can now be solved in period t to find:

$$U_t = \Gamma_{1t} X_t + \Gamma_{2t} E_t + C_{4t} \tag{4.24}$$

and

$$e_t = H_{1t} X_t + H_{2t} E_t + C_{5t} \tag{4.25}$$

The method described above solves the finite period problem, for an arbitrary terminal period (T). Note that the rule matrices are time subscripted because, in general, the rule in any period will be influenced by the terminal period. To find the solution to the infinite period problem we search for the limit to the backward recursion procedure where the rule matrices become independent of period T. The backward recursion procedure is repeated until the rule matrices converge to a stable value. The convergence is governed by the same conditions required to solve a rational expectation model using the Blanchard-Kahn technique. A necessary condition is that the number of eigenvalues outside the unit circle must be equal to the number of jumping variables.

In any period we can then find the solution to the model by knowing the state variables inherited and the constants which are derived using rules to accumulate all future exogenous variables, as well as future constant policy responses. In this procedure, it is a simple matter to set the policy rules to zero at each step of the iteration. The jumping variable rules summarised in the H matrices ensure that the solution is on the unique stable manifold of the model, given the cumulated future values of the constants derived from the future path of all exogenous variables.

The procedure developed in this section is substantially faster than the other algorithms discussed at the beginning of this chapter, except Blanchard-Kahn, because it is based on a linearized system. It has the added advantage of allowing various simulation exercises to be performed without requiring the recalculation of the rule matrices. Once these are calculated they can be used to simulate any shocks to exogenous variables or initial conditions.

5. Conclusion

This paper has summarised the techniques frequently used for solving rational expectations models. It has also developed, in detail, an algorithm for solving rational expectations models when governments are assumed to act strategically in a multicountry setting. In the case of solving non-linear models, the Fair-Taylor technique is preferred over the Multiple-Shooting technique on the basis of computational cost and speed. It is also our experience that when non-linearities are of little importance, it is worth using a linear approximation to the non-linear model which can be solved very quickly using either the Blanchard-Kahn solution or the MSG algorithm.

REFERENCES

Begg, D. (1982) The Rational Expectations Revolution in Macroeconomics: Theories and Evidence. Johns Hopkins University Press, Baltimore.

Blanchard, O. and C. Kahn (1980) "The Solution of Linear Difference Models Under Rational Expectations." Econometrica 48, pp.305-311.

Buiter, Willem and R. Marston (1985) International Economic Policy Coordination. Cambridge University Press, Cambridge.

Chow, E.C. and P.S. Reny (1985) "On Two Methods for Solving and Estimating Linear Simultaneous Equations under Rational Expectations", Journal of Economic Dynamics and Control, 9, pp.63-75.

Fair, R. and J. Taylor (1983) "Solution and Maximum Likelihood Estimation of Dynamic Non-Linear Rational Expectations Models", Econometrica 51(4), pp.139-178.

Haas, R. and P. Masson (1986) "Minimod: Specification and Simulation Results" IMF Staff Papers, 33, 4, pp.722-767.

Ishii, N., McKibbin, W. and J. Sachs (1985) "The Economic Policy Mix, Policy Cooperation and Protectionism: Some Aspects of Macroeconomic Interdependence Among the United States, Japan and Other OECD Countries". Journal of Policy Modelling 7(4), pp.533-572.

Kydland, F. and E. Prescott (1977) "Rules Rather Than Discretion: The Inconsistency of Optimal Plans". Journal of Political Economy 85, pp.473-492.

Lipton, D., J. Poterba, J. Sachs and L. Summers (1982) "Multiple Shooting in Rational Expectations Models". Econometrica 50, pp.1329-1333.

McKibbin, W. and J. Sachs (1986) "Coordination of Monetary and Fiscal Policies in the OECD". National Bureau of Economic Research Working Paper No. 1800, Forthcoming in Jacob Frenkel (Ed) International Aspects of Fiscal Policy. University of Chicago Press.

Minford, P., Agenor, P. and E. Nowell (1986) "A New Classical Econometric Model of the World Economy" Economic Modelling 3, 3, pp.154-174.

Oudiz, G. and J. Sachs (1985) "International Policy Coordination in Dynamic Macroeconomic Models" in Buiter and Marston (1985), pp.274-330.

Sachs, J. and W. McKibbin (1986) "Macroeconomic Policies in the OECD and LDC External Adjustment". Forthcoming in Van Wijnbergen Sweder, and Francis Colaco (eds.) International Capital Flows and the Developing Countries.

Sheffrin, S. (1983) Rational Expectations. Cambridge University Press, Cambridge.

Taylor, J.B. (1985) "Rational Expectations Models in Macroeconomics". in Kenneth J. Arrow and Seppo Honkapohja Frontiers of Economics. Basil Blackwell Ltd., Oxford, pp.391-425.

Taylor, J.B. (1986) "The Treatment of Expectations in Large Multicountry Models" Brookings Conference on Empirical Macroeconomics for Interdependent Economies: Where Do We Stand?